ON THE CONSISTENCY OF THE DEFINABLE
TREE PROPERTY ON ℵ₁

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ABSTRACT. In this paper we prove the equiconsistency of "Every ω_1-tree which is first order definable over $(H_{\omega_1}, \varepsilon)$ has a cofinal branch" with the existence of a Π_1^1 reflecting cardinal. We also prove that the addition of MA to the definable tree property increases the consistency strength to that of a weakly compact cardinal. Finally we comment on the generalization to higher cardinals.

1. INTRODUCTION

A well known result of Aronszajn is the existence of an Aronszajn tree on $\aleph_1$, i.e. there exist an ω_1-tree with no uncountable branch. The construction uses the axiom of choice and therefore does not give a definable such tree.

Sierpinski [12] and Kurepa [8] proved that if Ramsey theorem holds for κ then κ is a strong limit cardinal. Erdős [4] proved that such a κ is inaccessible. They also provided counterexamples to Ramsey theorem on small cardinals. These counterexamples explicitly used a well-ordering of $P(\kappa)$. The proof raises the question whether we can find a definable counterexample.

In a straightforward generalization of Aronszajn's proof, Specker proved the existence of Aronszajn trees for every succesor κ^+ s.t $\kappa^{<\kappa} = \kappa$. This raised the question whether the GCH was needed for this result. Mitchell and Silver [11] have proved that the tree property on $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. Magidor and Shelah [10] have proved the consistency of the tree property on $\aleph_{\omega+1}$ from the existence of very large cardinals.

Mitchell's forcing for the tree property works well if one tries to obtain the tree property on two non-consecutive cardinals (e.g. $\aleph_2$ and $\aleph_4$). However, his methods fail to prove the consistency of the tree property on $\aleph_2$ and $\aleph_3$ together. By a result of Magidor this is indeed a substential difficulty since the consistency strength of the tree property on $\aleph_2$ and $\aleph_3$ together is much higher then a weakly compact (e.g. it implies the existence of $0^\sharp$). Abraham [1] has proved the consistency of the tree property on both $\aleph_2$ and $\aleph_3$ from a supercompact cardinal and a weakly compact cardinal above it. This result was extended by Cummings and Foremann [2] which proved the consistency of the tree property on all $\aleph_n$'s from many supercompacts.

Kunen, and Shelah and Harrington [5] considered the consistency strength of obtaining Lebesgue measurability of projective sets together with Martin's axiom. They proved the equiconsistency of this theory with the existence of weakly compact cardinals. This shows that adding Martin's axiom to projective measurability increases the consistency strength.

In this paper we consider the consistency strength of the definable tree property on $\aleph_1$, i.e. the existence of a model in which every ω_1-tree which is first order definable (with parameters) over $(H_{\omega_1}, \varepsilon)$, has a cofinal branch. Our proof method answers also a related question regarding definable counterexamples to Ramsey theorem on $\aleph_1$.

In section (2) we define the exact meaning of a definable $\kappa-$tree, and study some variations used in this paper. In section (3) we define the Π_1^1 reflecting cardinals and derive an extension property which is similar to the extension property of weakly compact cardinal. We also bound the consistency strength of the existence of a Π_1^1 reflecting cardinal by the existence of a Mahlo cardinal. In section (4) we show that by forcing with the well known Levy collapse of a Π_1^1 reflecting cardinal κ to $\aleph_1$, we obtain a model of the definable tree property. We also prove that in this model a definable Ramsey theorem on $\aleph_1$ holds. In section (5) we show

that our assumptions on κ are necessary by proving that if the definable tree property on $\aleph_1$ holds, then

$$L \models \aleph_1 \text{ is a } \Pi_1^1 \text{ reflecting cardinal.}$$

In section (6) we prove the equiconsistency of the definable tree property on $\aleph_1+$ MA with the existence of a weakly compact cardinal. This is done by exploiting the methods of Kunen and Shelah and Harrington [5]. We also comment that adding any reasonable failure of GCH does not add to the consistency strength. Finally in section (7) we comment on the consistency of the definable tree property on higher cardinals with GCH. We describe a forcing to get the definable tree property on all $\aleph_n$'s together with GCH by using ω many Π_1^1 reflecting cardinals.

2. Definable $\kappa-$trees

In this section we define various notions of definable $\kappa-$trees and study the relationship between these notions. First define the usual notion of a $\kappa-$tree.

Definition 2.1.   • A tree is a partially ordered set$(T, <_T)$ such that for any $t \in T$ the set $\{s \in T | s <_T t\}$ of predecessors of t is well-ordered under $<_T$, and there is a root $r \in T$ such that for any $t \in T$, such that $t \neq r$, $r <_T t$.
  • The α'th level of T denoted by T_α is the set of elements of T whose set of $T-$predecessors has order type α.
  • A tree $(T, <_T)$ is a $\kappa-$tree if $|T| = \kappa$ and for every α $|T_\alpha| < \kappa$.
  • A branch is a $<_T$ linearly ordered subset of T.
  • A cofinal branch is a branch which intersects every level of T.

Next we would like to give a definition of a definable $\kappa-$tree. Three different notions naturally arise

Definition 2.2.   • A $\kappa-$tree is *definable in the strict sense if its underlying set is κ, and $<_T$ is $\mathbf{\Sigma}_\omega\left((H_\kappa, \in)\right)$.*
  • *A $\kappa-tree$ is* definable in the wide sense *if its underlying set T and $<_T$ are both $\mathbf{\Sigma}_\omega\left((H_\kappa, \in)\right)$ and T has definable cardinality κ, i.e. there is a bijection $f : \kappa \leftrightarrow T$ which is $\mathbf{\Sigma}_\omega\left((H_\kappa, \in)\right)$.*
  • *A $\kappa-$tree is* definable in the very wide sense *if its underlying set T and $<_T$ are both $\mathbf{\Sigma}_\omega\left((H_\kappa, \in)\right)$.*

Obviously being definable in the wide sense implies being definable in the very wide sense, and being definable in the strict sense implies being definable in the wide sense. The following proposition states that being definable in the wide sense is almost equivalent to being definable in the strict sense.

Proposition 2.1. *If $(T, <_T)$ is a $\kappa-$tree definable in the wide sense then there is a tree $(\kappa, <_{T'})$ isomorphic to $(T, <_T)$ which is definable in the strict sense.*

Proof :
  Let f be the definable bijection $f : \kappa \rightarrow T$, and let $\psi(x, y, z)$ be the definition of $<_T$ from the parameter z. Now define $<_{T'}$ by $\alpha <_{T'} \beta$ iff $\psi(f(\alpha), f(\beta), z)$. The rest trivially follows. $\square$

  Using this proposition we will not distinguish between the strict and the wide notions of definability. Also notice that if there is a well-ordering of H_κ which is $\mathbf{\Sigma}_\omega\left((H_\kappa, \in)\right)$ then the last two notions coincide. However in general this is not the case, and we will distinguish between definability in the very wide sense and definability in the strict sense which we will adopt as the definition of definability. So from now on, a definable $\kappa-$tree is a tree definable in the strict sense.

3. Π_1^1 reflecting cardinals

  Let κ be a cardinal. We say that κ is Π_n^m reflecting, if κ is inaccessible and for every $A \subseteq V_\kappa$ definable over V_κ (with parameters) and for every Π_n^m sentence Φ, such that

$$(V_\kappa, \varepsilon, A) \models \Phi$$

there is an $\alpha < \kappa$ such that

$$(V_\alpha, \varepsilon, A \cap V_\alpha) \models \Phi.$$

The Π_n^m reflecting cardinals are a lightface analog of the Π_n^m indescribable cardinals. However they have a much weaker consistency strength. If κ is Π_1^1 reflecting, then an easy consequence is the fact that κ is the $\kappa'th$ inaccessible cardinal, since inaccessibility is a Π_1^1 property, and for each $\alpha < \kappa$ being the $\alpha'th$ inaccessible is also a Π_1^1 property. Next we prove the following lemma:

Lemma 3.1. *Suppose κ is a Mahlo cardinal then for every m, n,*
$\{\alpha < \kappa | \alpha$ is a Π_n^m reflecting cardinal$\}$ is stationary.

Proof :
Let κ be Mahlo, and let C be a club. We shall find a $\mu \in C$ such that μ is a Π_n^m reflecting cardinal. Let $e : (\Phi, \psi, a) \to \kappa$ be an enumeration of triples (Φ, ψ, a) s.t. Φ is a Π_n^m sentence, $\psi(x, a)$ is a first order formula in the free variable x, $a \in V_\kappa$. Also assume without loss of generality that $a \in V_{e(\Phi,\psi,a)}$ and $e(\Phi, \psi, a) < rank(a)^\flat$, where $\alpha^\flat$ is the least inaccessible above α, and that e is $1 - 1$ and onto κ. For each triple (Φ, ψ, a) define $g(e(\Phi, \psi, a))$ by:

$$g(e(\Phi, \psi, a)) = \begin{cases} \text{The least } \rho \in C \text{ such that } e(\Phi, \psi, a) < \rho \text{ and} \\ (V_\rho, \varepsilon, S) \models (x \in S \text{ iff } \psi(x, a)) \wedge \Phi, \\ \text{if there is such a } \rho. \\ \\ e(\Phi, \psi, a) + 1 \text{ otherwise.} \end{cases} \tag{3.1}$$

$g : \kappa \to \kappa$ satisfies for every α, $\alpha < g(\alpha)$ by its definition. Since κ is Mahlo the set of inaccessibles is stationary so there is an inaccessible $\mu \in C$ s.t. $g''\mu \subseteq \mu$. We claim that μ is Π_n^m reflecting. By definition μ is inaccessible. Suppose that $S \subseteq V_\mu$ is defined by $x \in S$ iff $V_\kappa \models \psi(x, a)$. Assume that $(V_\mu, \varepsilon, S) \models \Phi(S)$. Since $\mu \in C$ there is a $\rho \in C$ s.t. $(V_\rho, \varepsilon, S \cap \rho) \models \Phi$ and $S \cap \rho$ is defined by $\psi(x, a)$. Hence $g(e(\Phi, \psi, a))$ is defined by the first case, since $g''\mu \subseteq \mu$ and $e(\Phi, \psi, a) < \mu$. Now by the definition of e, we obtain that $g(e(\Phi, \psi, a)) < \mu$. Hence Φ reflects to $\left(V_{g(e(\Phi,\psi,a))}, \varepsilon, S \cap g(e(\Phi, \psi, a))\right)$. $\square$

The following theorem is the analog of the Hanf-Scott theorem on the indescribability of weakly compact cardinals.

Definition 3.1. κ has the extension property iff for every n and for every $A \subseteq V_\kappa$ first order definable over V_κ with parameters from V_κ, there is a transitive ser X, and $A^X \subseteq X$ such that $\kappa \in X$ and $(V_\kappa, \varepsilon, A) \prec_n \left(X, \varepsilon, A^X\right)$.

Remark. One can show that κ has the extension property if and only if for every n there is a transitive Σ_n-elementary end extension of V_κ. containing κ, that is, $\kappa \in X$ and $(V_\kappa, \varepsilon) \prec_n (X, \varepsilon)$.

Note that the fact that κ has the extension property is Σ_1^1.

Theorem 3.2. *A cardinal κ is Π_1^1 reflecting iff κ is inaccessible and has the extension property.*

The proof is identical to the proof of the Hanf-Scott theorem (see [7] pp. 59-60).
Proof :
Assume that κ is inaccessible and has the extension property. Let $A \subseteq V_\kappa$ be definable in V_κ. Let Φ be a Π_1^1 formula, say of the form $\forall Y \phi(Y)$, where $\phi(Y)$ is first order formula in the predicates Y and A, and assume that $(V_\kappa, \varepsilon, A) \models \Phi$. Let n be large enough so that the sentence $\exists \alpha \left(\forall Y \in V_{\alpha+1} \left((V_\alpha, \varepsilon, A \cap V_\alpha) \models \phi(Y)\right)\right)$ is Σ_n in $L_\varepsilon(A)$, the ambient language of $(V_\kappa, \varepsilon, A)$. Now let (X, ε, A^X) be given by the extension property, that is,

$$(V_\kappa, \varepsilon, A) \prec_n (X, \varepsilon, A^X) \tag{3.2}$$

where X is transitive and $\kappa \in X$. Therefore $A^X \cap V_\kappa = A$. Also note that $V_{\kappa+1}^X \subseteq V_{\kappa+1}$ and $V_\kappa^X = V_\kappa$ (for n large enough). Now since $(V_\kappa, \varepsilon, A) \models \Phi$ it follows that

$$(X, \varepsilon, A^X) \models \forall Y \in V_{\kappa+1} \left((V_\kappa^X, \varepsilon, A^X \cap V_\kappa^X) \models \phi(Y)\right)$$

and thus

$$(X, \varepsilon, A^X) \models \exists \alpha \left(\forall Y \in V_{\alpha+1} \left((V_\alpha, \varepsilon, A^X \cap V_\alpha) \models \phi(Y)\right)\right).$$

Hence by (3.2)

$$(V_\kappa, \varepsilon, A) \models \exists \alpha((V_\alpha, \varepsilon, A \cap V_\alpha) \models \Phi).$$

Hence

$$(V_\alpha, \varepsilon, A \cap V_\alpha) \models \Phi.$$

Thus κ is Π_1^1 reflecting.

For the other direction let κ be a Π_1^1 reflecting cardinal, and fix some $n < \omega$. Let σ be a Π_1^1 formula expressing the failure of the extension property relative to n, that is there is no transitive Σ_n-elementary

extension X of V_κ containing κ (see the above remark). Let τ be a Π_1^1 formula expressing the inaccessibility of κ. Let

$$C = \{\alpha < \kappa | (V_\alpha, \varepsilon) \prec_n (V_\kappa, \varepsilon)\}.$$

C is a club subset of κ which is Σ_{n+1} definable in V_κ . Let ψ be the formula expressing the fact that C is a club. Therefore,

$$(V_\kappa, \varepsilon, C) \models \sigma \wedge \tau \wedge \psi.$$

By Π_1^1 reflection there exists an α such that

$$(V_\alpha, \varepsilon, C \cap \alpha) \models \sigma \wedge \tau \wedge \psi.$$

Hence α is an inaccessible cardinal below κ which is a limit point of C. Hence $\alpha \in C$, so $(V_\alpha, \varepsilon) \prec_n (V_\kappa, \varepsilon)$. Let $X_0 = V_\alpha \cup \{V_\alpha\}$ and construct an elementary submodel $(X', E) \prec (V_\kappa, \varepsilon)$, containing X_0 of cardinality α. Let (X', E) be the Skolem hull of X_0 inside (V_κ, ε). Let (X, ε) be the transitive collapse of (X', E). It follows that $(V_\alpha, \varepsilon) \prec_n (X, \varepsilon)$, and $V_\alpha \in X$ hence

$$(V_\alpha, \varepsilon) \models \neg \sigma$$

and this is a contradiction. $\square$

We finish this section with the following observation on Π_1^1 reflecting cardinals:

Lemma 3.3. *Let κ be a Π_1^1 reflecting cardinal. Let T be a $\kappa-$tree definable in the very wide sense, then T has a cofinal branch.*

Proof :

Note that since κ is inaccessible $V_\kappa = H_\kappa$. Let T be a definable κ-tree definable in the very wide sense. Let n be large enough such that the assertion "$\forall \alpha\ T$ has a cofinal branch of length α." is $\boldsymbol{\Sigma}_n$ over $(V_\kappa, \varepsilon, T)$. By theorem 3.2 there is a transitive structure (X, ε, T) such that

$$(V_\kappa, \varepsilon, T) \prec_n (X, \varepsilon, T^X). \tag{3.3}$$

Since (for n large enough) $V_\kappa^X = V_\kappa \in X$, it follows that $T^X \cap V_\kappa^X = T$. Now, since T is a κ-tree it follows that

$$(V_\kappa, \varepsilon, T) \models \forall \alpha\ T \text{ has a cofinal branch of length } \alpha.$$

Therefore,

$$\left(X, \varepsilon, T^X\right) \models \forall \alpha\ T^X \text{ has a branch of length } \alpha.$$

Since $\kappa \in X$ we see that $\left(X, \varepsilon, T^X\right) \models T^X$ has a branch b of length κ. Because $T^X \cap V_\kappa = T$, this branch b is really a cofinal branch through T. $\square$

4. The forcing construction

In this section we describe the forcing construction and prove that the extended model satisfies the tree property for ω_1-trees first order definable over $(H_{\omega_1}, \varepsilon)$. Let κ be a Π_1^1 reflecting cardinal in V. Let

$$\mathbf{P} = \text{ Coll } (\omega, < \kappa)$$

be the Levy collapse of κ to ω_1 and for every $\alpha < \kappa$ let

$$\mathbf{P}_\alpha = \text{ Coll } (\omega, < \alpha) \tag{4.1}$$

be an initial segments of the forcing. Let G be a $\mathbf{P}$ generic filter. Our main theorem is

Theorem 4.1. $V[G] \models$ *"every definable ω_1-tree T has a cofinal branch". Moreover if $V = L$ then $V[G] \models$ "every ω_1-tree T definable in the wide sense over $(H_{\omega_1}, \varepsilon)$ has a cofinal branch".*

Define the following definable partition relation :

Definition 4.1. $\aleph_1 \overset{def}{\to} (\aleph_1)_\alpha^m$ iff every partition of $[\aleph_1]^m$ into α sets which is first order definable over $(H_{\omega_1}, \varepsilon)$ (with parameters in H_{ω_1}), has a homogeneous set of size $\aleph_1$.

In order to prove theorem 4.1 we shall first prove a definable Ramsey theorem in $V[G]$.

Lemma 4.2. $V[G] \models \aleph_1 \overset{def}{\to} (\aleph_1)_2^2$

Proof :

Let $F : [\aleph_1]^2 \to 2$ be a definable function in $V[G]$, defined by

$$F(\{\alpha, \beta\}) = i \iff \Phi(x, \alpha, \beta, i) \tag{4.2}$$

where Φ is a $\mathbf{\Sigma}_n$ formula relativized to $(H_{\omega_1})^{V[G]}$, and the parameter x can be taken as a real, that is, a function $x : \omega \to \omega$. By the κ-c.c. of the Levy collapse for every real $x \in V[G]$ there is an $\varepsilon < \kappa$ such that $x \in V[G_\varepsilon]$, where G_ε is an initial segment of the generic, which is generic for $\mathbf{P}_\varepsilon = $ Coll $(\omega, < \varepsilon)$. Moreover by a result of Solovay (see [7] proposition 10.21) $V[G] = V[G_\varepsilon][H]$, and H is generic for the Levy collapse Coll $(\omega, < \kappa)$. By the homogeneity of the Levy collapse every set of ordinals definable in $V[G]$ with parameters in $V[G_\varepsilon]$ is definable in $V[G_\varepsilon]$, and for every formula Ψ we can compute another formula Φ such that:

$$V[G] \models \Psi(x, \alpha, \beta, i) \iff V[G_\varepsilon] \models \Phi(x, \alpha, \beta, i) \tag{4.3}$$

for all $\alpha, \beta < \kappa$ and $i \in \{0, 1\}$. Fix such an ε. Let $\tilde{\sigma}$ be a $\mathbf{P}_\varepsilon$ name for x. Now we define a κ-tree T which is definable in V_κ. For each $\alpha < \kappa$ let

$$\tilde{h} \in T_\alpha \iff$$
$$\tilde{h} \text{ is a } \mathbf{P}_\varepsilon \text{ name for a function from } \alpha \text{ to } \{0, 1\} \text{ and}$$
$$\exists \mu_0 \forall \mu \exists \delta > \mu \forall \beta < \alpha \forall p \in \mathbf{P}_\varepsilon \tag{4.4}$$
$$p \Vdash_{\mathbf{P}_\varepsilon} \Phi(\tilde{\sigma}, \beta, \mu_0, \tilde{h}(\beta)) \iff p \Vdash_{\mathbf{P}_\varepsilon} \Phi(\tilde{\sigma}, \beta, \delta, \tilde{h}(\beta))$$

and define $T = \cup_{\alpha < \kappa} T_\alpha$. We shall write $\tilde{h}_\alpha$ to denote that $\tilde{h}_\alpha \in T_\alpha$.
The ordering of T is

$$\tilde{h}_\alpha \leq \tilde{h}_\beta \iff \Vdash_{\mathbf{P}_\varepsilon} \tilde{h}_\alpha \subset \tilde{h}_\beta \tag{4.5}$$

The tree T is a κ-tree. To prove this first observe that for every $\alpha < \kappa$ $T_\alpha \neq \emptyset$ since for every α there are at most $2^{2^{\alpha + |\mathbf{P}_\varepsilon|}}$ many $\mathbf{P}_\varepsilon$ names for such functions, and therefore we are partitioning κ into less than κ many subsets according to the possible values of $\langle F(\{\beta, \delta\}) : \beta < \alpha \rangle$. Secondly $|T_\alpha| \leq 2^\alpha < \kappa$. T is definable in the very wide sense in H_κ by (4.4). Therefore by lemma (3.3) T has a cofinal branch $\left\langle \tilde{h}_\alpha : \alpha < \omega_1 \right\rangle$. Work now in $V[G]$. Let $\tilde{h}_\alpha(G_\varepsilon)$ denote the realization of the name $\tilde{h}_\alpha$ in $V[G_\varepsilon]$. Let $h = \cup_{\alpha < \kappa} \tilde{h}_\alpha(G_\varepsilon)$, h is a function from $\kappa = \aleph_1^{V[G]}$ to $\{0, 1\}$. Define

$$A_\alpha = \{\alpha < \gamma < \aleph_1 | \forall \beta < \alpha F(\{\beta, \gamma\}) = h(\beta)\} \tag{4.6}$$

then for every α, $|A_\alpha| = \aleph_1$ by the definition of $\tilde{h}_\alpha$. Moreover $\langle A_\alpha : \alpha < \aleph_1 \rangle$ is a decreasing sequence of sets. We construct H_0 by induction on $\alpha < \aleph_1$. Let $\beta_0 = 0$. For each α let $\gamma_\alpha = \sup \{\beta_i | i < \alpha\}$. Let $\beta_\alpha = \min A_{\gamma_\alpha}$. Let $H_0 = \{\beta_\alpha | \alpha < \aleph_1\}$. By the definition of H_0, for every $\alpha < \beta \in H_0$ we have $F(\{\alpha, \beta\}) = h(\alpha)$. Let l be minimal such that $|h^{-1}(l) \cap H_0| = \aleph_1$. Now $H = \{\alpha \in H_0 | h(\alpha) = l\}$ is the homogeneous set. $\quad \square$

Note that the proof really gives the following consequence

$$V[G] \models \aleph_1 \overset{def}{\to} (\aleph_1)^2_{\aleph_0}.$$

If $V = L$, or there is a definable well-ordering of H_κ in V, then the same proof yields the following:

Lemma 4.3. *Assume $V = L$. Let $h, A \in V[G]$ be such that $|A| = \aleph_1$, and $h : [A]^2 \to 2$ is a partition of $[A]^2$ into two parts, where both A and h are first order definable (with parameters) over $(H_{\omega_1}, \varepsilon)^{V[G]}$. Then there is a $B \subseteq A$ homogeneous for h and $|B| = \aleph_1$.*

To derive theorem 4.1 we follow the proof that a weakly compact cardinal has the tree property (see lemma 29.6 of [6]), replacing the partition property, of a weakly compact cardinal with the definable partition lemma (4.2).

proof of theorem 4.1

Let $T = (\aleph_1, <_T)$ be a definable tree on $\aleph_1$, i.e. there is a formula $\Psi(\alpha, \beta, z)$ such that

$$\alpha <_T \beta \iff (H_{\omega_1}, \varepsilon) \models \Psi(\alpha, \beta, z). \tag{4.7}$$

We extend the partial tree ordering $<_T$ into a linear ordering as follows:

$\underline{\alpha \prec \beta}$ iff

(i) $\alpha <_T \beta$ or

(ii) α, β are $<_T$ incomparable and if ζ is the first level where the predecessors of $\alpha, \beta, \alpha_\zeta, \beta_\zeta$ are distinct then $\alpha_\zeta < \beta_\zeta$.

$\prec$ is first order definable in $(H_{\omega_1}, \varepsilon)$ using the definition of $<_T$. Now define a partition of $[\aleph_1]^2$ by

$$F(\{\alpha, \beta\}) = 1 \iff \alpha < \beta \text{ agrees with } \alpha \prec \beta. \tag{4.8}$$

Since both $\prec$ and $<_T$ are definable F is definable as well. Hence there is $H \subset \aleph_1$ which is homogenous for F, with $|H| = \aleph_1$. Let

$$B = \{x \in \aleph_1 |\ |\{\alpha \in H | x <_T \alpha\}| = \aleph_1\}. \tag{4.9}$$

Since every level is countable there are members of B of every level. If we'll prove that any two members of B are $<_T$ comparable, then B will be the $\aleph_1$-branch. Let $x, y \in B$ be $<_T$ incomparable elements. Assume, without loss of generality, that $x \prec y$. Since both x, y have $\aleph_1$ many $<_T$ succesors in H we can find $\alpha, \beta, \mu \in H$ such that $\alpha <_T \beta <_T \mu$, $x <_T \alpha, \mu$ and $y <_T \beta$. By the definition of $\prec$ we get $\alpha \prec \beta$ and $\mu \prec \beta$. Thus $F(\{\alpha, \beta\}) = 1$ and $F(\{\beta, \mu\}) = 0$, contradicting the fact that H is homogeneous for F. Finally note that if we force over L theorem 4.1 can be strengthened to trees definable in the very wide sense. The proof is identical using lemma 4.3 instead of 4.2. $\square$

5. The lower bound

In this section we prove that the definable tree property implies the consistency of a Π_1^1 reflecting cardinal. In this section let $\aleph_1$ denote $\aleph_1^V$.

Theorem 5.1. *If $\aleph_1$ has the definable tree property then*

$$L \models \aleph_1 \text{ is a } \Pi_1^1 \text{ reflecting cardinal.}$$

First we prove that

$$L \models \aleph_1 \text{ is inaccessible.} \tag{5.1}$$

Assume that $\aleph_1$ is not inaccessible in L then there is an $x \in \omega^\omega$ such that $\aleph_1 = \aleph_1^{L[x]}$ (see [7] proposition 11.5). However inside $L[x]$ there is a special Aronszajn tree T which is definable from the well ordering of $(H_{\omega_1}, \varepsilon)^{L[x]}$, which is itself Σ_1 definable over $(H_{\omega_1}, \varepsilon)^{L[x]}$ [3]. However T cannot have a cofinal branch in V since this implies $\aleph_1^{L[x]} < \aleph_1$. Similarly, by relativization, for every real x $\aleph_1$ is inaccessible in $L[x]$.

Next we prove that $\aleph_1$ is a Π_1^1-reflecting cardinal in L. The proof is based on an idea from [9]. We define a tree using the Σ_n definable power set of $\aleph_1$. Note that since $\aleph_1$ is inaccessible in L, by (5.1), we have $(H_{\aleph_1^V})^L = L_{\aleph_1^V} = \left(V_{\aleph_1^V}\right)^L$. From a cofinal branch in the tree we define an ultrafilter on the Σ_n definable subsets of $\aleph_1$, and construct an "ultrapower" of $L_{\aleph_1}$, using only functions Σ_n definable over $(H_{\omega_1}, \varepsilon)$. Note that this is a definable tree in the strong sense since there is a definable well-oredering of the underlying set. Let $\langle A_\alpha | \alpha < \aleph_1 \rangle$ be a definable enumeration of $P^L(\aleph_1) \cap \Sigma_n$ of order type $\aleph_1$. Define a tree T of functions by

$$f \in T \iff f : \tau \to \{0, 1\}, \tau < \aleph_1 \text{ and } |\cap_{\alpha < \tau} A_\alpha^{f(\alpha)}| = \aleph_1$$

where $A^0 = A$, and $A^1 = \aleph_1 \backslash A$. The ordering on T is $f <_T g$ iff $f \subseteq g$. Since $\aleph_1$ is inaccessible in L the tree is an ω_1-tree. Since the truth of Σ_n formulas is a Σ_{n+1} definable, the tree T is Σ_k definable over $(H_{\omega_1}, \varepsilon)$, for some k. Hence by the definable tree property it has a cofinal branch producing a function $b : \aleph_1 \to \{0, 1\}$. b defines an ultrafilter U on $P^L(\aleph_1) \cap \Sigma_n$ by

$$A_\alpha \in U \iff b(\alpha) = 0$$

and this ultrafilter is countably complete on $P^L(\aleph_1) \cap \Sigma_n$. The "ultrapower" is now defined by

$$f \in ult(L_{\aleph_1}, U) \iff \begin{array}{l} f : \aleph_1 \to L_{\aleph_1}, f \in L \\ \text{and } f \text{ is } \Sigma_n \text{ definable over } (H_{\omega_1}, \varepsilon) \end{array} \tag{5.2}$$

$$f \equiv g \iff \{\alpha | f(\alpha) = g(\alpha)\} \in U \tag{5.3}$$

and

$$fEg \iff \{\alpha | f(\alpha) \in g(\alpha)\} \in U. \tag{5.4}$$

The "ultrapower" is wellfounded by the completeness of the ultrafilter and there is a Σ_n embedding $j : L_{\aleph_1} \prec_n ult(L_{\aleph_1}, U)$, by the proof of Łos theorem. However since $L_{\aleph_1} \models V = L$ if n is large enough

$ult(L_{\aleph_1}, U) \models V = L$, and hence its transitive collapse is really L_α for some $\alpha > \aleph_1$. Therfore $L_{\aleph_1} \prec_n L_\alpha$. This gives the desired extension property, and by the equivalence of the extension property and Π_1^1 reflection $\aleph_1$ is Π_1^1 reflecting in L. Finally by relativizing we obtain that for every real x, $\aleph_1$ is Π_1^1 reflecting in $L[x]$. $\square$

6. The definable tree property and Martin's axiom

In this section we investigate the consistency strengh of the Definable Tree Property on $\aleph_1$ together with Martin's Axiom and large continuum. The exposition is based on the Shelah-Harrington paper [5]. The main theorem is the following :

Theorem 6.1. *Let $\aleph_0 < \lambda$ be a cardinal satisfying $\lambda^{<\lambda} = \lambda$. The following are equiconsistent :*

1. *The definable tree property on $\aleph_1$ $+$ MA $+$ $2^{\aleph_0} = \lambda$*
2. *$\aleph_1$ is weakly compact cardinal in L.*

For the $2 \Rightarrow 1$ direction, we use lemma 5.1 which proves that the definable tree property implies that for every real x $\aleph_1^{L[x]} < \aleph_1$. Now we finish by the following result from [5]:

Theorem 6.2. *Assume MA then either there exists a real x such that $\aleph_1^{L[x]} = \aleph_1$, or $\aleph_1$ is weakly compact in L.*

The proof of the other direction follows closely Kunen's forcing for the consistency of "$MA+$ Every set in L(R) is Lebesgue measureable $+2^{\aleph_0} = \lambda$". The model we use is Kunen's model, and we only prove that the definable tree property holds in that model. For completeness we present the full proofs of Kunen's basic observations, as presented in [5]. Let κ be a weakly compact cardinal.

Lemma 6.3. *If B is a complete Boolean Algebra with the $\kappa-$c.c., and if $X \subseteq B$ satisfies $|X| < \kappa$, then there is a complete subalgebra $\bar{B}$, s.t. $X \subseteq \bar{B}$ and $|\bar{B}| < \kappa$.*

Proof :

Since B satisfies the $\kappa-$c.c., and $\kappa^{<\kappa} = \kappa$, there is B' a complete subalgebra of B such that $X \subseteq B'$ and $|B'| \leq \kappa$. Without loss of generality assume that $B' \subseteq \kappa$. Let D be the set of maximal antichains of B'. $D \subseteq [\kappa]^{<\kappa}$. By Π_1^1 reflection there is an $\alpha < \kappa$ s.t. $B' \cap \alpha$ is $< \alpha$ complete and $D \cap [\alpha]^{<\alpha}$ is the set of maximal antichains of $B' \cap \alpha$. Therefore $B' \cap \alpha$ is a complete subalgebra of B. If we choose $\alpha > \sup(X)$ then $X \subseteq B' \cap \alpha$. $\square$

Lemma 6.4. *If P_0, P_1 are two complete Boolean algebras with $\kappa-$c.c., then $P_0 \times P_1$ has the $\kappa-$c.c.*

Proof :

Let $\langle < p_\alpha^0, p_\alpha^1 > | \alpha < \kappa \rangle$ be a sequence of elements of $P_0 \times P_1$. Define $F : [\kappa]^2 \to 2 \times 2$. $F(\{\alpha, \beta\})(i) = 0$ iff p_α^i, p_β^i are compatible. A size κ homogeneous set for F gives a size κ set of pairwise compatible elements, since by the $< \kappa-$c.c of P_i the homogeneous color is $< 0, 0 >$. $\square$

Assume that $\nu < \lambda = \lambda^{<\lambda}$. To obtain the model we iterate λ many times $\kappa-$c.c. forcings of size $< \lambda$ using finite support, the same way this is done for MA ([14]). By lemma (6.4) we can assume, without loss of generality, that each forcing appears λ many times in the iteration. Denote the iteration by B. To prove that every $\aleph_1$ tree which is ordinal definable from a real has a branch in V^B, we first prove that Ramsey theorem for $\aleph_1$ holds for $L(R)$ partitions. Then we finish the same way as in the proof of theorem (4.1).

Let $F : [\aleph_1]^2 \to 2$ be definable in V^B from a real x, and ordinal parameters. By lemma (6.3) x is generic for a countable subalgebra $\bar{B}$. Moreover since each forcing in the iteration appears unboundedly many times we can assume , without loss of generality, that $B/\bar{B}$ is an homogeneous iteration over $V^{\bar{B}}$ which is built the same way as B is built over V. Therefore every value of F is decided inside $V^{\bar{B}}$. Since $|\bar{B}| < \kappa$ we can construct (in V) a κ tree of $\bar{B}$ names of possible values of F, the same way as we have built it in lemma (4.2). Now by the weak compactness of κ we can find a branch through that tree. Finally we use the branch to obtain the homogeneous set the same as we have done in lemma (4.2).

Note that since κ is weakly compact we are able to find homogeneous sets for every $L(R)$ coloring, and not just first order definable over H_{ω_1}. $\square$

Finally we remark that like in Solovay's proof of Lebesgue measurability [13], just adding the negation of the Continuum Hypothesis does not add to the consistency strength. We have to notice that the product

of collapsing κ to ω_1 and then adding λ Cohen reals satisfies the $\kappa-$c.c. The product is also homogeneous enough. Moreover every real belongs to a small generic extension, which is complemented by a homogeneous forcing. Hence the same situation as in theorem 4.1 still holds, and the argument there can be carried out.

7. The definable tree property on higher cardinals

This section contains several remarks regarding the definable tree property on cardinals above $\aleph_1$. By Mitchel's and Silver's results [11], the tree property on $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. However it is well known that assuming GCH, or even $\lambda^{<\lambda} = \lambda$, we have a λ^+ special Aronszajn tree. This is proved by the same construction as Aronszajn's construction on $\aleph_1$, using the fact that under this hypothesis there is a universal linear order of cardinality λ.

We will prove that assuming a Π_1^1 reflecting cardinal the definable tree property on $\aleph_2$ is consistent with GCH. Then we consider the property of having the definable tree property on succesive cardinals. We will generalize our forcing argument to prove that if it is consistent that there are ω Π_1^1 reflecting cardinals then it is consistent to have the definable tree property on $\langle \aleph_i : 1 \leq i < \omega \rangle$.

Theorem 7.1. *Assume κ is a Π_1^1 reflecting cardinal in L. Let $P = Coll(\aleph_1, < \kappa)$. Let G be P generic over L. Then*

$$L[G] \models GCH \text{ and } \aleph_2 \text{ has the definable tree property.}$$

Proof :

The proof is identical to the $\aleph_1$ case. Using the homogeneity of the Levy collapse, and the fact that κ is Π_1^1 reflecting. We just build a tree of possible values for the definable function. Then using the definabiltiy of the function we obtain definability of the tree and thus we can use the branch to prove the definable version of Ramsey theorem for $\aleph_2$. The fact that GCH holds in $V[G]$, is proved in the usual way, by assuming GCH in the ground model. $\square$

To obtain the definable tree property on all $\aleph_n$'s from a sequence of ω Π_1^1 reflecting cardinals, $\kappa_0 < \kappa_1 < \ldots$ just iterate the forcings $Coll(\aleph_n, < \kappa_n)$ with finite support. The proof that the definable tree property holds for every n, is a straightforward generalization and will not be given here.

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